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LETTER TO THE EDITOR

Stochastic Loewner evolution and Dyson's circular ensembles

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Abstract

Stochastic Loewner evolution (SLE$_\kappa$) has been introduced as a description of the continuum limit of cluster boundaries in two-dimensional critical systems. We show that the problem of N radial SLEs in the unit disc is equivalent to Dyson's Brownian motion on the boundary of the disc, with parameter $\beta = 4/\kappa$. As a result, various equilibrium critical models give realizations of circular ensembles with β different from the classical values of 1, 2 and 4 which correspond to symmetry classes of random $U(N)$ matrices. Some of the bulk critical exponents are related to the spectrum of the associated Calogero–Sutherland Hamiltonian. The main result is also checked against the predictions of conformal field theory.

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1. Introduction

Recently, a new method for understanding the scaling limit of conformally invariant two-dimensional critical systems has been introduced by Schramm [1] and developed by Lawler, Schramm and Werner [2] (LSW). This is known as stochastic Loewner evolution (SLE). It relies on the fact that many such systems may be realized geometrically in terms of sets of random curves, whose statistics can be described by a stochastic dynamical process. SLE is the continuum limit of this process.

There is in fact a continuous family of SLE processes, labelled by a real parameter $\kappa \geqslant 0$. Different values of κ are supposed to correspond to different universality classes of critical phenomena. For example, for $4 \leqslant \kappa \leqslant 8$ they describe the perimeters of the Fortuin–Kastelyn clusters of the Q-state Potts model with $4 \geqslant Q \geqslant 0$, while for $2 \leqslant \kappa \leqslant 4$ they describe the graphs of the high-temperature expansion of the $O(n)$ model with $-2 \leqslant n \leqslant 2$ (dual to

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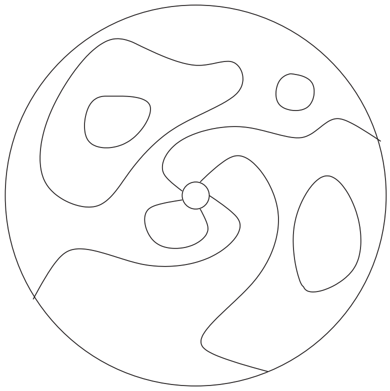

Figure 1. The geometrical set-up. N (here $= 3$) open curves connect the boundaries at $r = \epsilon$ and R of the annulus, intersecting the outer boundary at points $\{R\,\mathrm{e}^{\mathrm{i}\theta_j}\}$. No other open curves are allowed to intersect $r = R$, but they may intersect $r = \epsilon$, as well as there being any number of closed loops (except when they carry zero weight, as for $n = 0$).

the boundaries of critical Ising spin clusters for $n = 1$), as well as the external boundaries of FK clusters. This correspondence has so far been proved rigorously only in a few cases [3]. However, if the continuum limit of the lattice curves exists and is conformally invariant, it must be described by SLE [2].

Under these assumptions, LSW [2] have rederived many of the known results for two-dimensional critical behaviour which have been found by less rigorous approaches such as Coulomb gas methods and conformal field theory, as well as establishing some new ones. One aspect of the connection with conformal field theory has recently been pointed out by Bauer and Bernard [4] and Friedrich and Werner [5].

The particular setting we consider in this note is as follows: consider a critical system in a disc of radius R, with a puncture at the origin of radius ϵ, in the limit when R is much larger than ϵ and the lattice spacing a. Suppose there are exactly N open curves connecting the inner and outer boundaries. In addition, there are no open curves which begin and end on the outer boundary, see figure 1. For example, these curves could be mutually avoiding self-avoiding walks [6], or the external boundaries of percolation clusters (both $\kappa = \frac{8}{3}$), or the boundaries of critical Ising spin clusters ($\kappa = 3$.) (In these last two cases we assume that the ensemble is conditioned so as to satisfy the above.) Another example, conjectured to correspond to $\kappa = 4$, is when the curves are the level lines of a two-dimensional crystalline surface at the roughening transition, and there is a screw dislocation of strength N located at the origin. As long as $\kappa \leqslant 4$ it is known [7] that, in the continuum limit, these curves are simple, that is, they self-intersect with probability zero. For the same reason, the positions at which the curves intersect the outer boundary, labelled by complex numbers $R\,\mathrm{e}^{\mathrm{i}\theta_j}$, are well defined for $\kappa \leqslant 4$.

Our main result is that, for $R \gg \epsilon$, the joint probability density function (p.d.f.) of these points is given by Dyson's circular ensemble [8]

$$P_{\mathrm{eq}}(\{\theta_j\}) \propto \prod_{1\leqslant j<k\leqslant N} |\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\theta_k}|^{\beta} \tag{1}$$

with $\beta = 4/\kappa$. Our argument proceeds by showing that the SLE process appropriate for this situation contains the Brownian process invented by Dyson [9], whose equilibrium distribution is given by (1), with time being asymptotically proportional to $\ln(R/\epsilon)$.

The distribution (1) is known to describe the statistics of the eigenvalues of random unitary matrices in the orthogonal, unitary and symplectic ensembles for $\beta = 1$, 2 and 4 respectively.

Our arguments thus provide simple physical realizations of this ensemble for other values of $\beta > 1$, for example $\beta = \frac{3}{2}$ (self-avoiding walks) and $\beta = \frac{4}{3}$ (Ising spin cluster boundaries.)

We also check (1) against the predictions of conformal field theory (CFT). There is a subtle factor of $\frac{1}{2}$ in the exponent which we elucidate. Dyson's process is known to be related by a similarity transformation to the quantum Calogero–Sutherland model [10]. We point an interesting connection between the dilation operator $D \equiv L_0 + \bar{L}_0$ of CFT and the Calogero–Sutherland Hamiltonian.

It turns out that the eigenvalues of this Hamiltonian may, with suitable boundary conditions, correspond to bulk scaling exponents of the models with $\kappa > 4$, an example being the one-arm exponent computed by LSW [11].

2. Multiple SLEs

Our arguments are based on an N-particle generalization of radial SLE. A single radial SLE describes the continuum limit of a curve in the unit disc $U := \{z : |z| < 1\}$ which begins on the boundary at time $t = 0$ and ends up at the origin 0 as $t \to \infty$. Let $K(t)$ be the hull of the process up to time t (for $\kappa \leqslant 4$ this is just the set of points on the curve). There is a conformal mapping $g_t : U\backslash K(t) \to U$, such that $g_t(0) = 0$ and $g_t'(0) > 0$. LSW argue that $g_t(z)$ may be chosen so as to satisfy an evolution equation

$$\dot{g}_t(z) = -g_t(z)\frac{g_t(z) + \mathrm{e}^{\mathrm{i}\sqrt{\kappa}B(t)}}{g_t(z) - \mathrm{e}^{\mathrm{i}\sqrt{\kappa}B(t)}} \tag{2}$$

where $B(t)$ is a standard one-dimensional Brownian process with $\mathbf{E}[B(t)^2] = t$. Note that time has been reparametrized so that $g_t'(0) = \mathrm{e}^t$. Equation (2) is the standard form of radial SLE, which maps the trace of the SLE into the point $\mathrm{e}^{\mathrm{i}\sqrt{\kappa}B(t)}$ on the boundary, but for our purposes it is more convenient to consider $\hat{g}_t(z;\theta) \equiv g_t(z)\,\mathrm{e}^{\mathrm{i}(\theta-\sqrt{\kappa}B(t))}$, which maps the trace into $\mathrm{e}^{\mathrm{i}\theta}$, and satisfies[2]

$$\mathrm{d}\hat{g}_t(z;\theta) = -\hat{g}_t(z;\theta)\frac{\hat{g}_t(z;\theta) + \mathrm{e}^{\mathrm{i}\theta}}{\hat{g}_t(z;\theta) - \mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}t - \mathrm{i}\hat{g}_t(z;\theta)\sqrt{\kappa}\,\mathrm{d}B(t). \tag{3}$$

Now consider N SLEs which start from distinct points $\{\mathrm{e}^{\mathrm{i}\theta_j}\}$ on the boundary, with $1 \leqslant j \leqslant N$. Let $K_j(t)$ be the hull of the jth SLE. For $\kappa \leqslant 4$ these are segments of non-intersecting simple curves. Let $G_t^{(N)}(z)$ be a function which conformally maps $U\big\backslash \cup_{j=1}^N K_j(t)$ onto U, with $G_t^{(N)}(0) = 0$ and $G_t^{(N)\prime}(0) > 0$. Then we shall argue that $G_t^{(N)}(z)$ may be chosen to satisfy

$$\dot{G}_t^{(N)} = -G_t^{(N)}\sum_{j=1}^N \frac{G_t^{(N)} + \mathrm{e}^{\mathrm{i}\theta_j(t)}}{G_t^{(N)} - \mathrm{e}^{\mathrm{i}\theta_j(t)}} \tag{4}$$

where

$$\mathrm{d}\theta_j(t) = \sum_{k\neq j}\cot((\theta_j(t) - \theta_k(t))/2)\,\mathrm{d}t + \sqrt{\kappa}\,\mathrm{d}B_j(t) \tag{5}$$

and $B_j(t)$ are N independent Brownian motions, starting at the origin.

To see this, consider the infinitesimal transformation $G_{t+\mathrm{d}t}^{(N)} \circ \left(G_t^{(N)}\right)^{-1}$ and note that this may be obtained by allowing each SLE to evolve independently according to (3) over a time $\mathrm{d}t$:

$$G_{t+\mathrm{d}t}^{(N)} \circ \left(G_t^{(N)}\right)^{-1} = g_{\mathrm{d}t}^{(N)}(\theta_N(t)) \circ g_{\mathrm{d}t}^{(N-1)}(\theta_{N-1}(t)) \circ \cdots \circ g_{\mathrm{d}t}^{(1)}(\theta_1(t)). \tag{6}$$

[2] In the Ito convention there is an additional term $-\frac{\kappa}{2}\hat{g}\mathrm{d}t$ on the right-hand side. This disappears again after making the global rotation leading from (8) to (4).

During the evolution of the jth SLE, $G_t^{(N)}$ evolves according to (3), with $\theta = \theta_j(t)$ and $B(t) = B_j(t)$, but the other $\theta_k(t)$ with $k \neq j$ also evolve according to

$$\mathrm{d}\theta_k(t) = \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\theta_k(t)} + \mathrm{e}^{\mathrm{i}\theta_j(t)}}{\mathrm{e}^{\mathrm{i}\theta_k(t)} - \mathrm{e}^{\mathrm{i}\theta_j(t)}}\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_j(t) = \cot((\theta_k(t) - \theta_j(t))/2)\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_j(t). \qquad (7)$$

Thus, after evolving every SLE with $j = 1, \ldots, N$, we have

$$\mathrm{d}G_t^{(N)} = -G_t^{(N)} \sum_{j=1}^{N} \frac{G_t^{(N)} + \mathrm{e}^{\mathrm{i}\theta_j(t)}}{G_t^{(N)} - \mathrm{e}^{\mathrm{i}\theta_j(t)}}\,\mathrm{d}t - \mathrm{i}G_t^{(N)}\sqrt{\kappa}\sum_{j=1}^{N}\mathrm{d}B_j(t) \qquad (8)$$

where

$$\mathrm{d}\theta_j(t) = \sum_{k\neq j}\cot((\theta_j(t) - \theta_k(t))/2)\,\mathrm{d}t - \sqrt{\kappa}\sum_{k\neq j}\mathrm{d}B_k(t). \qquad (9)$$

It is now simpler to rotate the whole disc through an angle $\sqrt{\kappa}\sum_{j=1}^{N}\mathrm{d}B_j(t)$, after which we obtain (4) and (5) as claimed.

Equation (5) is the Dyson process [9]. It may be written as

$$\mathrm{d}\theta_j = -\frac{\partial V}{\partial\theta_j}\,\mathrm{d}t + \sqrt{\kappa}\,\mathrm{d}B_j(t) \qquad (10)$$

where $V \equiv -2\sum_{j<k}\ln|\sin((\theta_j - \theta_k)/2)|$. At late times, the distribution of the $\{\theta_j(t)\}$ tends towards an equilibrium p.d.f. at temperature $\kappa/2$:

$$P_{\mathrm{eq}}(\theta_1, \ldots, \theta_N) \propto \mathrm{e}^{-2V/\kappa} = \prod_{j<k}|\sin((\theta_j - \theta_k)/2)|^{4/\kappa} \propto \prod_{j<k}|\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\theta_k}|^{\beta} \qquad (11)$$

This is Dyson's circular ensemble [8], with

$$\beta = 4/\kappa. \qquad (12)$$

So far, the starting points $\{\theta_j\}$ have been taken to be fixed. Now consider an ensemble of these, generated by the configurations of some bulk critical ensemble in the interior of U (conditioned if necessary on the existence of exactly N curves). We argue that if this corresponds to a conformally invariant bulk system, the p.d.f. of the $\{\theta_j\}$ must be given by P_{eq}. Let $K(\{\theta_j\}, t)$ be the union of the N hulls up to time t, given that they start at the points $\{\mathrm{e}^{\mathrm{i}\theta_j}\}$. The expectation value of any observable may be taken by first conditioning on the subset $K(\{\theta_j\}, t')$, with $t' < t$. By conformal invariance, the distribution of $K(\{\theta_j\}, t) \setminus K(\{\theta_j\}, t')$ is the same as that of its image under $G_{t'}^{(N)}$, namely $K(\{\theta_j(t')\}, t - t')$. Averaging over $K(\{\theta_j\}, t')$ is equivalent to averaging over the $\{B_j\}$ up to time t'. Taking $t \to \infty$, we conclude that

$$P(\{\theta_j\}) = \mathbf{E}_{\{B_j(t''):t''\in[0,t']\}}[P(\{\theta_j(t')\})] \qquad (13)$$

that is $P(\{\theta_j\})$ is stationary under the process (5), and must therefore be equal to P_{eq}.

This is strictly valid only when the SLEs are allowed to reach the origin. To discuss the case when they reach only the circle $|z| = \epsilon$, it is helpful to map conformally the annulus to a cylinder of length $\ell \equiv \ln(R/\epsilon)$. The points $\mathrm{e}^{\mathrm{i}\theta_j}$ are now arrayed around one end of the cylinder. Since $\dot{G}_t^{(N)} = NG_t^{(N)}\big(1 + O\big(G_t^{(N)}\big)\big)$ as $G_t^{(N)} \to 0$, we see that as long as $\ell \gg 1$ the effect of the evolution is to reduce the length of the cylinder at a rate $\dot{\ell} = -N$. Meanwhile the points $\{\theta_j\}$ are moving according to (5). The approach of their distribution to equilibrium is expected to be exponentially fast with a rate constant $O(1)$. Thus, as long as $\ell \gg N$, we may apply the same argument as above, and deduce that the distribution of the $\{\theta_j\}$ is given by P_{eq}, with corrections suppressed by powers of ϵ/R. The same should apply on a lattice, as long as the spacing $a < \epsilon \ll R$.

3. Comparison with conformal field theory

The crucial assumption of conformal invariance made in deriving the above result would appear to be stronger than the analogous statement for $N = 1$. In particular, it is not clear from this point of view why invariance under the uniformizing transformation $G_t^{(N)}$, which assumes that the curves grow at the same rate, is to be chosen among other possibilities, although it appears to be the most natural one. For this reason, we have checked our main result (1) using methods of CFT. In this language we expect the joint p.d.f. to be given by the correlation function in the $O(n)$ conformal field theory

$$\left\langle \Phi_{a_1\ldots a_N}(0)\phi_{a_1}(R\,\mathrm{e}^{\mathrm{i}\theta_1})\ldots\phi_{a_N}(R\,\mathrm{e}^{\mathrm{i}\theta_N})\right\rangle \tag{14}$$

where $\phi_a(R\,\mathrm{e}^{\mathrm{i}\theta})$ is a boundary 1-leg operator carrying $O(n)$ index a, and Φ is a bulk N-leg operator. By choosing the a_j to be all different, we ensure that the curves all reach the origin without annihilating. In [12], it was conjectured that the operators ϕ_a correspond to Virasoro representations labelled by $(1, 2)$ in the Kac classification. These have a null state at level 2, and therefore their correlators satisfy second-order linear partial differential equations with respect to each of the θ_j (the BPZ equations [13]). The general solution for such an $N + 1$-point correlator is not however known. Instead, we may take the form (1) as an ansatz, and check whether it satisfies these equations. Even this is somewhat tedious, and we have carried it through only for $N = 2$. Alternatively, one may check whether (1) satisfies the fusion rules which follow from the BPZ equations. These determine the behaviour of the correlator (14) in the limits when (say) p of the θ_j approach each other. Suppose, for example, that $|\theta_j - \theta_k| = O(\delta)$ for $1 \leqslant j \leqslant p$ and $1 \leqslant k \leqslant p$, with $2 \leqslant p \leqslant N$. In the limit $\delta \to 0$ we may use the operator product expansion (OPE)

$$\prod_{j=1}^{p}\phi_{a_j}(\theta_j) \propto \delta^{x_p - p x_1}\phi_{a_1\ldots a_p}(\theta_1) \tag{15}$$

where $\phi_{a_1\ldots a_p}$ is the boundary p-leg operator, and x_p is its scaling dimension. Given that the 1-leg operator corresponds to $(1, 2)$, the fusion rules determine the allowed values of x_p which may occur on the right-hand side of (15): they are the scaling dimensions $h_{1,n+1}$ of the $(1, n + 1)$ operators, with $0 \leqslant n \leqslant p$ and $p - n$ even. Duplantier and Saleur [14] argued that the p-leg operator must in fact correspond to $n = p$. Using the Kac formula $h_{1,p+1} = p(2p + 4 - \kappa)/2\kappa$ then gives the exponent in (15) to be simply $p(p - 1)/\kappa$.

On the other hand, we may simply take the appropriate limit in the ansatz (1) to find a dependence on δ of the form $\prod_{1\leqslant j<k\leqslant p}\delta^{\beta} = \delta^{p(p-1)\beta/2}$. Comparing these two expressions apparently gives $\beta = 2/\kappa$, not $4/\kappa$ as found above. One may take further short-distance limits within (1): the results all consistent with Duplantier and Saleur [14] but only if $\beta = 2/\kappa$.[3]

The resolution of this paradox is as follows: the correlator (1) may be written in operator language as

$$\langle \Phi_N|\mathrm{e}^{-tD}|\{\theta_j\}\rangle \tag{16}$$

where $t = \ln(R/\epsilon)$, $D \equiv L_0 + \bar{L}_0$ is the generator of scale transformations and $|\{\theta_j\}\rangle$ is a boundary state. The usual formalism of CFT assumes that D is self-adjoint: this originates in the invariance of the bulk theory under inversions $z \to -1/z$. Alternatively, on the cylinder,

[3] We remark that the reason that the $N + 1$-point correlator (14) has such a simple form is that there is only a single term on the right-hand side of OPEs such as (15). This is because all the $O(n)$ indices on the left-hand side are different, and therefore the operators on the right-hand side transform according to the totally symmetric representation of $O(n)$. The other possible terms, with $n < p$, correspond to other representations, and could only arise if some of the a_j were equal.

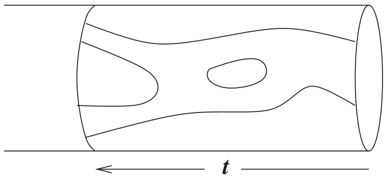

Figure 2. The transfer matrix of [15] on the cylinder. It keeps track of the positions of points where curves intersect the given time-slice, as well as their connectivity in the past (but not the future.) We distinguish between those points connected to the boundary at $t = 0$ ($r = R$), and those which close for times $t > 0$.

e^{-aD} is the continuum limit of the transfer matrix, which, for many lattice models, may also be chosen to be self-adjoint. But for the loop representation of the $O(n)$ model this is not the case: the only practicable transfer matrix which has been employed [15] acts, at a given time t, on the space spanned by a basis defined by the positions of the points at which the loops intersect the chosen time slice, together with their relative connections in the 'past', but not the 'future'. This asymmetry leads to a transfer matrix T which is not self-adjoint. In our case, the points at which the loops intersect a given time slice can either be connected back to the points $e^{i\theta_j}$ at $t = 0$, or to each other via loops which close in the past at times $t > 0$ (see figure 2). Let Π be the projection operator in the full space in which T acts, onto the subspace spanned by the boundary states $|\{\theta_j\}\rangle$. That is, Π traces over the positions of the points not connected to $t = 0$. Then $\tilde{T} \equiv \Pi T$ acts wholly within this subspace. Note that $\int \prod_j d\theta_j \langle\{\theta_j\}|$ is a left eigenstate of $\tilde{T}$ with unit eigenvalue.

As usual, we may also think of $\tilde{T}$ as acting on L^2 functions of the $\{\theta_j\}$ through

$$(\tilde{T}f)(\{\theta_j\}) = \int \prod_j d\theta'_j \langle\{\theta_j\}|\tilde{T}|\{\theta'_j\}\rangle f(\{\theta'_j\}) \tag{17}$$

so that $\tilde{T}^\dagger \mathbf{1} = 1$. Moreover, as $\ln(R/\epsilon) \to \infty$, the joint p.d.f. of the $\{\theta_j\}$ is given by the right eigenfunction of $\tilde{T}$ with the largest eigenvalue.

On the other hand, the Langevin equation (5) yields the Fokker–Planck equation $\dot{P} = \mathcal{L}P$ for the evolution of $P(\{\theta_j\}, t)$ where

$$\mathcal{L} = \sum_{j=1}^{N} \frac{\partial}{\partial\theta_j}\frac{\partial V}{\partial\theta_j} + \frac{\kappa}{2}\frac{\partial^2}{\partial\theta_j^2} \tag{18}$$

where $\mathcal{L}P_{eq} = 0$ and $\mathcal{L}^\dagger \mathbf{1} = 0$. Now the conformal mapping $G_t^{(N)} \propto e^{Nt}$ acts as a scale transformation near the origin, while its action on the unit disc is described by $\mathcal{L}$. We therefore conjecture that this is nothing but the continuous version of $\tilde{T}$, more precisely, $\Pi T^t \sim e^{at\mathcal{L}/N}$, acting on the subspace.

Now for any dynamics which satisfies detailed balance $\mathcal{L}$ is related to a self-adjoint operator H by a similarity transformation:

$$H = -P_{eq}^{-1/2}\mathcal{L}P_{eq}^{1/2}. \tag{19}$$

The ground state eigenfunction of H is then $P_{eq}^{1/2}$. This square root is the origin of the discrepancy between the CFT result and (1) with $\beta = 4/\kappa$: the self-adjoint operator e^{-Ht} is proportional to the scale transformation operator e^{-Dt} where $D = L_0 + \bar{L}_0$ of CFT, acting on the subspace spanned by the boundary states, that is $e^{-Ht/N} \propto \Pi\, e^{-Dt}\Pi^\dagger$. But it does not give the continuum limit of the correct transfer matrix. By assuming that this limit was self-adjoint, which is implicit in the standard formulation of CFT, we found, erroneously, the square root of the correct result (1) with $\beta = 4/\kappa$.

In the case of Dyson's Brownian motion the Hamiltonian H is that of the quantum Calogero–Sutherland model [10]

$$\mathcal{H} = -\frac{\kappa}{2}\sum_j \frac{\partial^2}{\partial\theta_j^2} + \frac{2-\kappa}{2\kappa}\sum_{j<k}\frac{1}{\sin^2(\theta_j-\theta_k)/2} - \frac{N(N-1)}{2\kappa}. \tag{20}$$

It is interesting to note that the adjoint operator $\mathcal{L}^\dagger = \mathrm{e}^{2V/\kappa}\mathcal{L}e^{-2V/\kappa}$ has the form

$$\mathcal{L}^\dagger = \frac{\kappa}{2}\sum_j \frac{\partial^2}{\partial\theta_j^2} + \sum_j\sum_{k\neq j}\cot((\theta_j-\theta_k)/2)\frac{\partial}{\partial\theta_j} \tag{21}$$

$\mathcal{L}^\dagger$ is the generator for a typical first-passage problem. For example, in the case $N = 2$, the probability $h(\theta_1, \theta_2; t)$ that the two particles have not met up to time t, given that they started from (θ_1, θ_2) satisfies $\partial_t h = \mathcal{L}^\dagger h$. In fact, with $\theta \equiv \theta_1 - \theta_2$, and rescaling $2t \to t$, this is just the equation derived by LSW [11], whose lowest non-trivial eigenvalue gives the one-arm exponent, related to the fractal dimension of FK clusters for $\kappa = 6$. Eigenfunctions of $\mathcal{L}^\dagger$ behave near $\theta = 0$ as θ^α where $\alpha = 0$ or $1 - 4/\kappa$. When $\kappa \leqslant 4$ the appropriate solution corresponds to $\alpha = 0$, or $h = 1$, consistent with the result that SLE is a simple curve, but when $\kappa > 4$ the solution is non-trivial. LSW [11] argue that the appropriate boundary condition at $\theta = 2\pi$ for the one-arm problem is $\partial h/\partial\theta = 0$, and that the solution is then $(\sin(\theta/4))^{1-4/\kappa}\,\mathrm{e}^{-\lambda t}$ with $\lambda = (\kappa^2 - 16)/32\kappa$. This is of course also an eigenvalue of the Calogero–Sutherland Hamiltonian (20), and it raises the question as to whether other bulk scaling dimensions are given by eigenvalues of Calogero–Sutherland systems with suitable boundary conditions.

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